CATEGORIES OF FRACTIONS REVISITED

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Abstract. The theory of categories of fractions as originally developed by Gabriel and Zisman [GZ67] is reviewed in a pedagogical manner giving detailed proofs of all statements. A weakening of the category of fractions axioms used by Higson [Hig90] is discussed and shown to be equivalent to the original axioms.

Contents

1. Introduction

Categories of fractions are the analogue in category theory to rings of fractions in ring theory: adjoining formal inverses for a certain class of morphisms yields a new category from an existing one. Due to the metamathematical nature of category theory however, the objectives are quite different and categories of fractions are a tool to construct new *mathematical theories* from existing ones. And in some cases these abstract constructions can be more useful than concrete (in the category-theoretical sense) descriptions of the categories under investigation. Furthermore, categories of fractions can be relevant for other general categorical constructions; the theory of Verdier localization in the framework of triangulated categories is an example.

Section 2 is mainly included for completeness; there, the concept of localization of a category is introduced and compared to taking a quotient category. Section 3 then gives a detailed account of the category of fraction axioms and their consequences; in particular, all proofs are presented in full. Section 4 goes on to study a weakening of the category of fraction axioms which was originally introduced by Higson in [Hig90] in the context of bivariant K-theory of C^*-algebras. It is shown that this weakening is equivalent to the usual set of axioms. This result is the only possibly new result of the present work. Finally, section 5 shows that a category of fractions is additive in case the original category is additive.

Some words about notation and terminology: in all commutative diagrams, the objects are simply denoted by fat dots "$\bullet$". Except in cases where a commutativity

statement is explicitly made, all diagrams commute. Idenitity morphisms are pictured as double lines " $=\!=\!=$ ". The words "isomorphism" and "monomorphism" are abbreviated respectively as "iso" and "mono". A split mono is a morphism which has a left inverse; it automatically is a mono. Domain and codomain of a morphism f are written as $\mathrm{dom}(f)$ and $\mathrm{cod}(f)$, respectively.

This article is a revised version of the first chapter of the author's Master thesis.

2. Localization of categories

In some contexts it may happen that we have a category $\mathcal{C}$ which is – in a sense depending on the situation – not well-behaved. For example, it might be that it is too hard to do concrete calculations, or it might be that $\mathcal{C}$ does not have some desired formal property. Then one can try to find a second category $\widehat{\mathcal{C}}$ which has the same objects as $\mathcal{C}$ together with a functor $\mathcal{C} \to \widehat{\mathcal{C}}$ which is the identity on objects, such that $\widehat{\mathcal{C}}$ is better-behaved and approximates $\mathcal{C}$ in some appropriate sense also depending on the situation. Then instead of working in $\mathcal{C}$ directly, one can transport the morphisms from $\mathcal{C}$ to $\widehat{\mathcal{C}}$ via the functor $\mathcal{C} \to \widehat{\mathcal{C}}$ and prove theorems about the morphisms in the well-behaved category $\widehat{\mathcal{C}}$. The price one has to pay is that in general some information about the structure of $\mathcal{C}$ is lost on the way.

Now there are at least two concrete ways to make this precise. The first one is the notion of a **quotient category**. Suppose we are given an equivalence relation $\sim$ on every morphism set $\mathcal{C}(A, B)$ which is preserved under composition, meaning that

$$\text{(eq. 1)} \qquad (f_1 \sim f_2) \Longrightarrow (f_1 g \sim f_2 g) \wedge (h f_1 \sim h f_2) \qquad \forall f_1, f_2, g, h$$

whenever these compositions are defined. Then composition of equivalence classes is well-defined and defines the quotient category $\mathcal{C}/\!\sim$ together with the canonical projection functor $\mathcal{C} \to \mathcal{C}/\!\sim$. Any kind of homotopy theory serves as a good example.

The second way is a concept familiar from ring theory called **localization**. Suppose we are given a category $\mathcal{C}$ and a subclass of morphisms called $\mathcal{W}$, which "morally" ought to be isos, but in $\mathcal{C}$ not necessarily all of them are; using the letter $\mathcal{W}$ is supposed to suggest a reading like "weak equivalence" [Hov99]. We try to turn all the morphisms in $\mathcal{W}$ into isos by adjoining formal inverses for them. More precisely, we are looking for a category $\widehat{\mathcal{C}} = \mathcal{C}[\mathcal{W}^{-1}]$ and a localization functor $Loc : \mathcal{C} \to \mathcal{C}[\mathcal{W}^{-1}]$, such that it has the following universal property:

(a) $Loc(w)$ is an iso for all $w \in \mathcal{W}$
(b) If $F : \mathcal{C} \to \mathcal{D}$ is any functor which also maps $\mathcal{W}$ to isos, then F factors uniquely over Loc as in the diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "Loc"] \arrow[dr, "F"'] & & \mathcal{C}[\mathcal{W}^{-1}] \arrow[dl, dashed, "!"] \\
& \mathcal{D} &
\end{tikzcd}$$

In case such a functor exists, the category $\mathcal{C}[\mathcal{W}^{-1}]$ is called the "localization of $\mathcal{C}$ with respect to $\mathcal{W}$". It serves as the desired approximation $\widehat{\mathcal{C}}$ to $\mathcal{C}$.

Since $\mathcal{C}[\mathcal{W}^{-1}]$ is defined via a universal property, it is certainly unique (up to a unique iso). Existence is the nontrivial part.

2.1. **Theorem.** *$\mathcal{C}[\mathcal{W}^{-1}]$ and Loc always exist.*

Proof. (from [GM03, III.2.2] and [GZ67, 1.1]). The category $\mathcal{C}[\mathcal{W}^{-1}]$ can be constructed in two steps: start with the category of paths – call it $\mathcal{P}(\mathcal{C}, \mathcal{W}^{-1})$ – which has as objects the objects of $\mathcal{C}$, and as morphisms finite strings $\langle l_1, \dots l_n \rangle$ of composable literals, where a literal l_k is either a morphism of $\mathcal{C}$ (including $\mathcal{W}$) or a formal inverse of a morphism in $\mathcal{W}$. Composition of morphisms is defined as concatenation of strings. For every object $A \in \mathcal{C}$, we also have the empty string $\langle\,\rangle_A$ which starts and ends at A and is the identity morphism of A in $\mathcal{P}(\mathcal{C}, \mathcal{W}^{-1})$. (As an alternative definition, $\mathcal{P}(\mathcal{C}, \mathcal{W}^{-1})$ could be described as the free category generated by the graph $\mathcal{C} \cup \mathcal{W}^{-1}$.)

There is a canonical map $\mathcal{C} \to \mathcal{P}(\mathcal{C}, \mathcal{W}^{-1})$ which is the identity on objects and maps every morphism of $\mathcal{C}$ to the corresponding single-literal string. This map already has the desired universal property (b). However, neither is this map a functor nor does it map $\mathcal{W}$ to isos. We can easily fix both of these issues by taking a quotient category of $\mathcal{P}(\mathcal{C}, \mathcal{W}^{-1})$ such that these properties hold. Therefore, introduce the equivalence relation $\sim$ on strings generated by closure under composition together with the elementary equivalences

(a) $\langle\,\rangle_A \sim \langle \mathrm{id}_A \rangle$
(b) $\langle g, f \rangle \sim \langle gf \rangle$ (assuming the composition exists)
(c) $\langle w, w^{-1} \rangle \sim \langle\,\rangle_{\mathrm{cod}(w)}\,, \quad \langle w^{-1}, w \rangle \sim \langle\,\rangle_{\mathrm{dom}(w)}$

Then it is clear that the induced map $Loc : \mathcal{C} \to \mathcal{P}(\mathcal{C}, \mathcal{W}^{-1})/\sim$ is a functor and maps $\mathcal{W}$ to isos.

As for universality, suppose we are given some functor $F : \mathcal{C} \to \mathcal{D}$ mapping $\mathcal{W}$ to isos. It induces a unique functor $\mathcal{P}(\mathcal{C}, \mathcal{W}^{-1}) \to \mathcal{D}$. This functor maps the above elementary equivalences to equalities, thus uniquely factors over the quotient category $\mathcal{P}(\mathcal{C}, \mathcal{W}^{-1})/\sim$. □

2.2. **Remark.** (a) For locally small $\mathcal{C}$, the localization $\mathcal{C}[\mathcal{W}^{-1}]$ need not be locally small. Even for the categories of fractions discussed in the next section, it may well happen that the localization has proper classes as the collections of morphisms between some pairs of objects. Showing that this does not happen in a concrete case seems to be a hard problem; one case where local smallness is known is for model categories, where localizing with respect to weak equivalences yields the locally small homotopy category (see [Hov99, p.7 and 1.2.10]).

(b) The canonical functor to a quotient category $\mathcal{C} \to \mathcal{C}/\sim$ is full by definition of $\mathcal{C}/\sim$. However, this is usually not true for a localization functor $\mathcal{C} \to \mathcal{C}[\mathcal{W}^{-1}]$.

3. Categories of fractions

In all diagrams dealing with categories of fractions, a wiggly arrow $\rightsquigarrow$ denotes a morphism in $\mathcal{W}$, while a straight arrow $\longrightarrow$ is any morphism of $\mathcal{C}$.

In certain situations, $\mathcal{C}[\mathcal{W}^{-1}]$ can be described much more explicitly which implies a large gain of control over the structure of this category. We say that the pair $\mathcal{W} \subseteq \mathcal{C}$ allows a **calculus of left fractions**, if the following conditions are satisfied:

(L0) $\mathcal{W}$ contains all identity morphisms and is closed under composition. In other words, $\mathcal{W} \subseteq \mathcal{C}$ is a subcategory containing all objects.

(L1) Given any $w \in \mathcal{W}$ and an arbitrary morphism f with $\mathrm{dom}(f) = \mathrm{dom}(w)$, we can find $w' \in \mathcal{W}$ and some morphism f' with $\mathrm{cod}(f') = \mathrm{cod}(w')$, such that the diagram

$$\begin{tikzcd} \bullet \arrow[r, squiggly, "w"] \arrow[d, "f"'] & \bullet \arrow[d, "f'"] \\ \bullet \arrow[r, squiggly, "w'"'] & \bullet \end{tikzcd}$$

commutes.

(L2) Given $w \in \mathcal{W}$ and parallel morphisms f_1, f_2 such that $f_1 w = f_2 w$, there exists $w' \in \mathcal{W}$ such that $w' f_1 = w' f_2$.

$$\begin{tikzcd} \bullet \arrow[r, squiggly, "w"] & \bullet \arrow[r, bend left, "f_1"] \arrow[r, bend right, "f_2"'] & \bullet \arrow[r, squiggly, "w'"] & \bullet \end{tikzcd}$$

These conditions are exact analogues of the Ore conditions in the theory of noncommutative rings [Jat86, p. 3].

3.1. **Remark.** Note that (L0) is not an essential restriction: if (L1) and (L2) hold for some class of morphisms $\mathcal{W}$, then both also hold for the $\mathcal{C}$-subcategory generated by $\mathcal{W} \cup \{\mathrm{id}_A,\ A \in \mathcal{C}\}$, so we can replace $\mathcal{W}$ by this subcategory.

3.2. **Definition.** *Define a **roof** (f, w) between two objects $\mathrm{dom}(f)$ and $\mathrm{dom}(w)$ to be a diagram of the form*

$$\begin{tikzcd} & \bullet & \\ \bullet \arrow[ur, "f"] & & \bullet \arrow[ul, squiggly, "w"'] \end{tikzcd}$$

The way to think of a roof (f, w) is as being a formal "left fraction" $w^{-1} f$ (hence, "left roof" would actually be a more concise terminology than simply "roof"). Then (L1) states that it is possible to turn any right fraction $f w^{-1}$ into a left fraction $w'^{-1} f'$, since commutativity $w' f = f' w$ together with invertibility of w and w' implies $f w^{-1} = w'^{-1} f'$.

3.3. **Definition.** *Two roofs (f_1, w_1) and (f_2, w_2) are said to be **equivalent** if there are morphisms g and h forming a third roof $(g f_1, g w_1) = (h f_2, h w_2)$ as in the diagram*

$$\begin{tikzcd} & & \bullet & & \\ & \bullet \arrow[ur, "g"] & & \bullet \arrow[ul, "h"'] & \\ \bullet \arrow[ur, "f_1"] \arrow[urrr, "f_2"'] & & & & \bullet \arrow[ulll, squiggly, "w_1"] \arrow[ul, squiggly, "w_2"'] \arrow[uull, squiggly, bend right, "hw_2 = gw_1"'] \end{tikzcd}$$

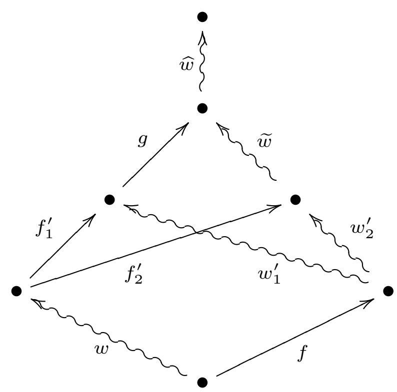
Note that it is not required that g or h be an element of $\mathcal{W}$, only the *composition* $g w_1 = h w_2$ has to be in $\mathcal{W}$. The equality of $(g f_1, g w_1)$ and $(h f_2, h w_2)$ is expressed by commutativity of the two squares in the diagram.

The goal of this section is to establish that equivalence classes of roofs form a category under the appropriate composition operation, and that this category is the localization of $\mathcal{C}$ with respect to $\mathcal{W}$. This will be done in a sequence of small steps.

The first is to show that turning any "right fraction" fw^{-1} into a left fraction roof $w'^{-1}f'$ is unique up to equivalence, which will let us define composition of equivalence classes of roofs later on.

3.4. **Lemma.** *Any two ways to choose f' and w' in (L1) define equivalent roofs.*

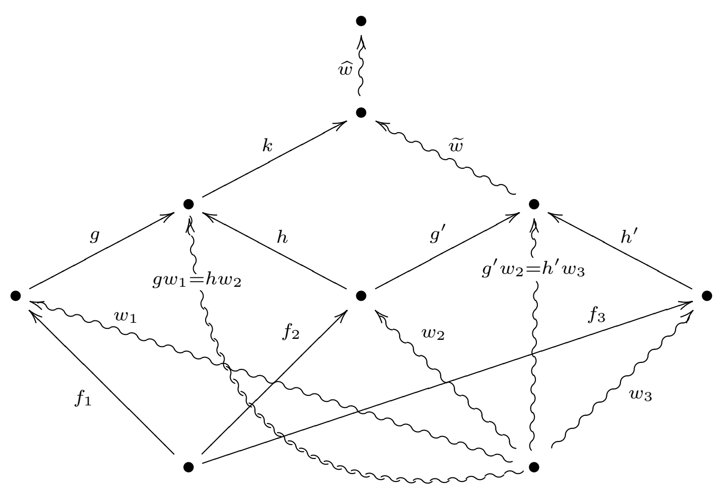
Proof. Imagine two possible choices (f_1', w_1') and (f_2', w_2') as in the partially commutative diagram



By (L1), g and $\widetilde{w}$ were chosen such that $gw_1' = \widetilde{w}w_2'$. This is not yet an equivalence of roofs, since, in general, $gf_1' \neq \widetilde{w}f_2'$. However, we do know that $gf_1'w = \widetilde{w}f_2'w$, so by (L2) we can choose $\widehat{w}$ such that $\widehat{w}gf_1' = \widehat{w}\widetilde{w}f_2'$. This makes (f_1', w_1') and (f_2', w_2') equivalent via $\widehat{w}g$ and $\widehat{w}\widetilde{w}$. $\square$

3.5. **Lemma.** *The equivalence of parallel roofs defined in (3.3) is an equivalence relation.*

Proof. Reflexivity and symmetry are obvious. For transitivity, suppose we are given an equivalence between (f_1, w_1) and (f_2, w_2), and one between (f_2, w_2) and (f_3, w_3), as in the partially commutative diagram



The commutativity conditions for the two equivalences are

$$gf_1 = hf_2, \quad gw_1 = hw_2; \qquad g'f_2 = h'f_3, \quad g'w_2 = h'w_3 \tag{eq. 2}$$

In the upper part of the diagram, k and $\widetilde{w}$ were obtained from (L1) applied to the two wiggly arrows $gw_1 = hw_2$ and $g'w_2 = h'w_3$. The corresponding commutativity assertion of (L1) then is $khw_2 = \widetilde{w}g'w_2$. By use of (L2), we can then find the drawn $\widehat{w}$ such that $\widehat{w}kh = \widehat{w}\widetilde{w}g'$. Together with the commuting squares relations (eq. 2), this means that the compositions $\widehat{w}kg$ and $\widehat{w}\widetilde{w}h'$ of the morphisms which go up along the sides implement an equivalence between (f_1, w_1) and (f_3, w_3). □

Under a closer look, this argument is actually a special case of the argument used to prove (3.4). In fact, we could have applied (3.4) directly to the two roofs (h, hw_2) and $(g', g'w_2)$ – both are (L1)-complements of the formal right fraction $\mathrm{id}_{\mathrm{dom}(w_2)} w_2^{-1}$ – and we would have been done.

3.6. **Remark.** One can also take a 2-categorical point of view which gives some more insight on the notion of equivalence of roofs. Define a (sort of) 2-category which has as objects the objects of $\mathcal{C}$ and as 1-morphisms the roofs in $\mathcal{C}$ with respect to $\mathcal{W}$. For a roof (f, w) we have $\mathrm{dom}((f, w)) = \mathrm{dom}(f)$ and $\mathrm{cod}((f, w)) = \mathrm{dom}(w)$. A 2-morphism from a roof (f_1, w_1) to a parallel roof (f_2, w_2) is defined to be a commutative diagram

$$\begin{tikzcd}
& \bullet \arrow[r, bend left] & \bullet & \\
\bullet \arrow[ur, "f_1"] \arrow[urr, "f_2"'] & & & \bullet \arrow[ull, squiggly, "w_1"] \arrow[ul, squiggly, "w_2"']
\end{tikzcd}$$

In other words, between every pair of objects we do not merely have a set of roofs as morphisms, but a whole category of them; this is very much in spirit of the 2-category of spans described in [ML98, XII.7]. This construction is only "sort of" a 2-category however, since a general horizontal composition of 2-morphisms does not seem to exst, while the composition of 1-morphisms can only be defined up to equivalence (see below).

Now the observation is that two roofs are equivalent if and only if they can be connected by a finite path of 2-morphisms, where each 2-morphism is either traversed from its domain to its codomain or in the reverse direction. To see this, note that the third roof (gf_1, hw_2) in the diagram of definition (3.3) is connected to each of the other two roofs by a 2-morphism. The other implication direction follows from the transitivity statement (3.5) and the fact that two parallel roofs connected by a single 2-morphism are equivalent. Hence the result (3.5) can also be reinterpreted as a connectivity statement about the category of parallel roofs between some pair of objects.

Lemma (3.4) also allows the definition of composition for equivalence classes of roofs:

3.7. **Definition.** *Given two roofs* (f_1, w_1) *and* (f_2, w_2)*, which are supposed to be composable – that is,* $\mathrm{dom}(w_1) = \mathrm{dom}(f_2)$ *– we define their composition as*

$$(f_2, w_2) \circ (f_1, w_1) \equiv (\widetilde{f} f_1, \widetilde{w} w_2)$$

where $\widetilde{f}$ and $\widetilde{w}$ in

$$\begin{tikzcd}
& & \bullet & & \\
& \bullet \arrow[ur, "\widetilde{f}"] & & \bullet \arrow[ul, squiggly, "\widetilde{w}"'] & \\
\bullet \arrow[ur, "f_1"] & & \bullet \arrow[ul, squiggly, "w_1"] \arrow[ur, "f_2"'] & & \bullet \arrow[ul, squiggly, "w_2"']
\end{tikzcd}$$

were obtained by means of (L1).

Thanks to (3.4), the equivalence class of $(\widetilde{f}, \widetilde{w})$ is unique, but then so is the equivalence class of $(\widetilde{f} f_1, \widetilde{w} w_2)$.

3.8. **Lemma.** *This composition does not depend on the equivalence class of either of the two roofs.*

Proof. For both pairs of roofs, it is sufficient to consider the case that they are connected by an elementary equivalence as described in (3.6). Thus suppose we are given the lower half of the diagram

$$\begin{tikzcd}
& & & \bullet & & & \\
& \bullet \arrow[r, "g_1"] & \bullet \arrow[ur, "\widetilde{f}"] & & \bullet \arrow[ul, squiggly, "\widetilde{w}"'] & \bullet \arrow[l, "g_2"'] & \\
\bullet \arrow[ur, "f_1'"] \arrow[urr, "f_1"'] & & & \bullet \arrow[ull, squiggly, "w_1'"] \arrow[ul, squiggly, "w_1"'] \arrow[ur, "f_2"] \arrow[urr, "f_2'"'] & & & \bullet \arrow[ull, squiggly, "w_2"] \arrow[ul, squiggly, "w_2'"']
\end{tikzcd}$$

which represents two pairs of elementarily equivalent roofs. After possible renamings $(f_1, w_1) \leftrightarrow (f_1', w_1')$ and $(f_2, w_2) \leftrightarrow (f_2', w_2')$, we can assume that g_1 goes from $\mathrm{cod}(f_1')$ to $\mathrm{cod}(f_1)$, while g_2 similarly points from $\mathrm{cod}(f_2')$ to $\mathrm{cod}(f_2)$.

Applying (L1) to the pair w_1, f_2 yields $\widetilde{f}$ and $\widetilde{w}$. Then $(\widetilde{f} f_1, \widetilde{w} w_2)$ is a possible roof representing the composition $(f_2, w_2) \circ (f_1, w_1)$. Similarly, $(\widetilde{f} g_1 f_1', \widetilde{w} g_2 w_2')$ is a possible roof representing the composition $(f_2', w_2') \circ (f_1', w_1')$. By commutativity, these roofs coincide, so in particular they are equivalent. $\square$

3.9. **Theorem.** *If $\mathcal{W} \subseteq \mathcal{C}$ admits a calculus of left fractions, then the category $\mathcal{C}[\mathcal{W}^{-1}]$ can be described as the category with the same objects as $\mathcal{C}$, morphisms equivalence classes of roofs, and composition as defined above. The localization functor $Loc : \mathcal{C} \to \mathcal{C}[\mathcal{W}^{-1}]$ is given by $f \mapsto (f, \mathrm{id})$.*

Proof. Associativity of composition follows from the diagram

$$\begin{tikzcd}
& & & \bullet & & & \\
& & \bullet \arrow[ur] & & \bullet \arrow[ul, squiggly] & & \\
& \bullet \arrow[ur] & & \bullet \arrow[ul, squiggly] \arrow[ur] & & \bullet \arrow[ul, squiggly] & \\
\bullet \arrow[ur] & & \bullet \arrow[ul, squiggly] \arrow[ur] & & \bullet \arrow[ul, squiggly] \arrow[ur] & & \bullet \arrow[ul, squiggly]
\end{tikzcd}$$

where the three lower roofs are those to be composed; the rest of the diagram is obtained by three applications of (L1). Then the large roof from the left to the right formed by composing the morphisms along the sides is a representative for the composition of the three lower roofs in both possible ways of bracketing. This shows associativity. Furthermore, the equivalence classes of the roofs $(\mathrm{id}, \mathrm{id})$ obviously function as identity morphisms.

Under *Loc*, the image of some $w \in \mathcal{W}$ is (w, id), and this image has as its inverse element the class of (id, w) since there is an obvious equivalence $(w, w) \sim (\mathrm{id}, \mathrm{id})$. For functoriality, *Loc* preserves identities by definition, and preserves composition by the diagram

$$\begin{tikzcd}
 & & \bullet & & \\
 & \bullet \arrow[ur, "g"] \arrow[dr, equal] & & \bullet \arrow[ul, equal] \arrow[dr, equal] & \\
\bullet \arrow[ur, "f"] & & \bullet \arrow[ur, "g"] & & \bullet
\end{tikzcd}$$

which says that the roof (gf, id) is a representative for the equivalence class of $(g, \mathrm{id}) \circ (f, \mathrm{id})$.

It remains to check universality. Suppose we have some functor $F : \mathcal{C} \to \mathcal{D}$ which maps $\mathcal{W}$ to isos. First we need to show that F uniquely extends to roofs. Clearly, any such extension has to map the roof (f, id) to $F(f)$. Similarly, since the class of (id, w) is the inverse of the class of (w, id), any such extension maps (id, w) to $F(w)^{-1}$. But now (f, w) is a representative of the composition $(\mathrm{id}, w) \circ (f, \mathrm{id})$, so we need $(f, w) \mapsto F(w)^{-1}F(f)$.

We still have to check that this assignment is well-defined on equivalence classes and that it is functorial. Consider an elementary equivalence of roofs as in (3.6)

$$\begin{tikzcd}
 & \bullet \arrow[rr, bend left, "g"] & & \bullet & \\
\bullet \arrow[ur, "f_1"] \arrow[urrr, "f_2"'] & & & & \bullet \arrow[ulll, squiggly, "w_1"] \arrow[ul, squiggly, "w_2"']
\end{tikzcd}$$

Then in $\mathcal{D}$ we have $F(w_2) = F(g)F(w_1)$, so $F(w_1)^{-1} = F(w_2)^{-1}F(g)$. Then the calculation

$$F(w_1)^{-1}F(f_1) = F(w_2)^{-1}F(g)F(f_1) = F(w_2)^{-1}F(f_2) \qquad \text{(eq. 3)}$$

shows that the equivalent roofs get mapped to identical morphisms in $\mathcal{D}$.

Functoriality follows by very similar reasoning. Given a pair of composable roofs together with their composition as in definition (3.7), it holds that

$$F(\widetilde{f})F(w_1) = F(\widetilde{w})F(f_2) \qquad \text{(eq. 4)}$$

so that we get

$$F(\widetilde{w})^{-1}F(\widetilde{f}) = F(f_2)F(w_1)^{-1} \qquad \text{(eq. 5)}$$

Applying first the functor and composing the roofs afterwards yields

$$\big(F(w_2)^{-1}F(f_2)\big) \circ \big(F(w_1)^{-1}F(f_1)\big) \qquad \text{(eq. 6)}$$

while for the other direction we end up with $F(\widetilde{w}w_2)^{-1}F(\widetilde{f}f_1)$, which coincides with (eq. 6) by (eq. 5) and functoriality of F. □

If $\mathcal{W} \subseteq \mathcal{C}$ satisfies (L0) (which is self-dual) and additionally the conditions (R1) and (R2), which are defined to be the duals of (L1) and (L2), then we say that $\mathcal{W}$ allows a **calculus of right fractions**, and of course also the dual theorem holds. If all five of the (L?) and (R?) conditions hold, we say that $\mathcal{W}$ admits a **calculus of left and right fractions**.

4. Weakening the requirements

In [Hig90], a notion of category of fractions is introduced which, on first sight, is seemingly weaker in its premises than the one presented above. While keeping (L0) and (L1), the axiom (L2) gets replaced by the weaker axiom

(L2') Denote by $\mathcal{W}_L$ the class of morphisms in $\mathcal{C}$ generated by $\mathcal{W}$ and all split monos in $\mathcal{C}$. Then given $w \in \mathcal{W}$ and parallel morphisms f_1, f_2 such that $f_1 w = f_2 w$, there exists $w' \in \mathcal{W}_L$ such that $w' f_1 = w' f_2$.

$$\begin{tikzcd} \bullet \arrow[r, squiggly, "w"] & \bullet \arrow[r, bend left, "f_1"] \arrow[r, bend right, "f_2"'] & \bullet \arrow[r, squiggly, "w'"] & \bullet \end{tikzcd}$$

4.1. **Proposition.** *Given $w' \in \mathcal{W}_L$, we can find $k \in \mathrm{Mor}(\mathcal{C})$ such that $kw' \in \mathcal{W}$.*

Proof. Let us consider the cases how w' might look like, one by one and in increasing order of difficulty. If already $w' \in \mathcal{W}$, we are done since we can take $k = \mathrm{id}_{\mathrm{cod}(w')}$. If $w' = m\widehat{w}$, where m is a split mono and $\widehat{w} \in \mathcal{W}$, we can take k to be a left-inverse of m, so we are done as well.

The only nontrivial type of situation occurs when elements of $\mathcal{W}$ come after split monos. The prototype for this situation is a morphism like $w' = \widehat{w}m$, with $\widehat{w} \in \mathcal{W}$. By assumption, the split mono m has a left inverse e, which means $em = \mathrm{id}$. Now apply (L1) to the pair $\widehat{w}, e$

$$\begin{tikzcd} \bullet \arrow[r, squiggly, "\widehat{w}"] \arrow[d, "e"] & \bullet \arrow[d, "k"] \\ \bullet \arrow[r, squiggly, "\widetilde{w}"'] \arrow[u, bend left, "m"] \arrow[ur, "w'"] & \bullet \end{tikzcd}$$

which gives the morphism k and some morphism $\widetilde{w} \in \mathcal{W}$. The commutativity assertion of (L1) states in this case $\widetilde{w}e = k\widehat{w}$, so after composing with m on the right we have $\widetilde{w} = \widetilde{w}em = k\widehat{w}m = kw' \in \mathcal{W}$.

Now for the general case. By definition of $\mathcal{W}_L$ and (L0), our w' is of the form

$$w' = w_n m_n \cdots w_1 m_1 \qquad \text{(eq. 7)}$$

where the m_j are split monos and $w_j \in \mathcal{W}$. Starting from the left, we can iteratively apply the previous argument and use (L0) to compose the morphisms in $\mathcal{W}$ to a single morphism in $\mathcal{W}$, until we have only a single morphism in $\mathcal{W}$ left. □

4.2. **Corollary.** *(a) Given (L0) and (L1), the assertions (L2) and (L2') are equivalent.*

(b) *Two roofs (f_1, w_1) and (f_2, w_2) are equivalent if and only if there is a diagram*

$$\begin{tikzcd}
 & & \bullet & & \\
 & \bullet \arrow[ur, "g"] & & \bullet \arrow[ul, "h"'] & \\
\bullet \arrow[ur, "f_1"] \arrow[urrr, "f_2"'] & & & & \bullet \arrow[ulll, squiggly, "w_1"] \arrow[ul, squiggly, "w_2"']
\end{tikzcd}$$

where now we only demand commutativity and $hw_2 \in \mathcal{W}_L$ (instead of $hw_2 \in \mathcal{W}$).

Proof. These are both immediate consequences of the previous proposition. □

4.3. **Remark.** As already noticed in [Hig90, 1.2.4], when (L0) holds the axiom (L1) is in fact equivalent to the variant where $w \in \mathcal{W}_L$:

(L1') Given any $w \in \mathcal{W}_L$ and an arbitrary morphism f with $\mathrm{dom}(f) = \mathrm{dom}(w)$, we can find $w' \in \mathcal{W}$ and some morphism f' with $\mathrm{cod}(f') = \mathrm{cod}(w')$, such that the diagram

$$\begin{tikzcd}
\bullet \arrow[r, squiggly, "w"] \arrow[d, "f"'] & \bullet \arrow[d, "f'"] \\
\bullet \arrow[r, squiggly, "w'"'] & \bullet
\end{tikzcd}$$

commutes.

Clearly (L1) is trivially implied by this. For the other implication direction, note that by (3.1) it is sufficient to show that (L1') holds if w is any split mono as in the diagram

$$\begin{tikzcd}
\bullet \arrow[r, "w"'] \arrow[d, "f"'] & \bullet \arrow[l, bend right, "e"'] \arrow[d, "fe"] \\
\bullet \arrow[r, equal] & \bullet
\end{tikzcd}$$

with e some left-inverse of w. Then by (L0) we have $w' \equiv \mathrm{id}_{\mathrm{cod}(f)} \in \mathcal{W}$, so together with $f' \equiv fe$ this does the job; commutavity $few = f$ holds since e is left-inverse to w.

5. Additive categories of fractions

Often the working mathematician has to deal with additive categories. In particular, he may want to do localization completely within the framework of additive categories. In other words, given an additive category $\mathcal{C}$ and a class of "moral isomorphisms" $\mathcal{W}$ in $\mathcal{C}$, is there an additive category $\mathcal{C}[\mathcal{W}^{-1}]$ and an additive localization functor $Loc : \mathcal{C} \to \mathcal{C}[\mathcal{W}^{-1}]$ which maps $\mathcal{W}$ to isos and is the universal additive functor with this property? And if yes, how can this localization be constructed?

For simplicity, we consider only the category of fractions case. Then, in fact, the localization constructed in (3.9) already *is* additive. Inituitively, the reason is that

one can find a "common denominator" for pairs of roofs representing parallel morphisms in $\mathcal{C}[\mathcal{W}^{-1}]$. The purpose of this section is to turn this intuitive explanation into a formal proof.

5.1. **Lemma.** *Given parallel roofs (f_1, w_1) and (f_2, w_2)*

$$\begin{tikzcd}
& \bullet & \bullet & \\
\bullet \arrow[ur, "f_1"] \arrow[urr, "f_2"'] & & & \bullet \arrow[ul, squiggly, "w_2"'] \arrow[ull, squiggly, "w_1"]
\end{tikzcd}$$

we can find some $w' \in \mathcal{W}$ and appropriate f_1', f_2' such that there are equivalences $(f_1, w_1) \sim (f_1', w')$ and $(f_2, w_2) \sim (f_2', w')$.

Proof. By applying (L1) to the pair w_1, w_2, we obtain a diagram

$$\begin{tikzcd}
& & \bullet & & \\
& \bullet \arrow[ur, "g"] & & \bullet \arrow[ul, squiggly, "\widetilde{w}"'] & \\
\bullet \arrow[ur, "f_1"] \arrow[urrr, "f_2"'] & & & & \bullet \arrow[ul, squiggly, "w_2"'] \arrow[ulll, squiggly, "w_1"]
\end{tikzcd}$$

which commutes only in the sense that $gw_1 = \widetilde{w}w_2$. In particular, the roofs (f_1, w_1) and $(gf_1, \widetilde{w}w_2)$ are equivalent. Similarly, (f_2, w_2) is equivalent to $(\widetilde{w}f_2, \widetilde{w}w_2)$. Thus we have identified $w' \equiv \widetilde{w}w_2$ as a common denominator. $\square$

Given any two roofs, we may assume that their second components coincide after applying the procedure outlined in the proof. Then we can define an addition operation simply by using the addition we have available in $\mathcal{C}$ as

$$(f_1, w_1) + (f_2, w_2) \equiv (gf_1 + \widetilde{w}f_2, \widetilde{w}w_2) \qquad \text{(eq. 8)}$$

Thanks to (3.4), the equivalence class of the right-hand side does not depend on the particular choices for $(g, \widetilde{w})$.

5.2. **Lemma.** *The class of the sum only depends on the classes of the summands and not on the particular representatives.*

Proof. Still using the same notation, it is sufficient to consider an elementary equivalence (see (3.6)) between (f_1, w_1) and some (f_1', w_1'):

$$\begin{tikzcd}
& \bullet \arrow[r, bend left, "h"] & \bullet & \\
\bullet \arrow[ur, "f_1'"] \arrow[urr, "f_1"'] & & & \bullet \arrow[ul, squiggly, "w_1"'] \arrow[ull, squiggly, "w_1'"]
\end{tikzcd}$$

Then taking the common denominator of (f_1, w_1) and (f_2, w_2) as in the previous proof yields the diagram

$$\begin{tikzcd}
& & & \bullet & \\
\bullet \arrow[rr, bend left, "h"] & & \bullet \arrow[ur, "g"] & & \bullet \arrow[ul, squiggly, "\widetilde{w}"'] \\
\bullet \arrow[u, "f_1'"] \arrow[urr, "f_1"] \arrow[urrrr, "f_2"'] & & & & \bullet \arrow[u, squiggly, "w_2"'] \arrow[ull, squiggly, "w_1"'] \arrow[ullll, squiggly, "w_1'"]
\end{tikzcd}$$

Now the sum of (f_1, w_1) and (f_2, w_2) is the class of

$$(gf_1 + \widetilde{w} f_2, \widetilde{w} w_2) \tag{eq. 9}$$

while the sum of (f_1', w_1') and (f_2, w_2) is the class of

$$(ghf_1' + \widetilde{w} f_2, \widetilde{w} w_2) \tag{eq. 10}$$

which coinides with (eq. 9) by commutativity of the diagram. □

5.3. **Theorem.** *Suppose $\mathcal{C}$ is additive and allows a calculus of left fractions with respect to $\mathcal{W}$. Then the category of fractions $\mathcal{C}[\mathcal{W}^{-1}]$ is additive.*

Proof. A category is additive if it is preadditive, has a zero object, and has a biproduct for every pair of objects.

It was already shown how to add equivalence classes of roofs and that this operation is well-defined. Addition of roof equivalence classes is obviously associative and commutative since the one in $\mathcal{C}$ is. Neutral elements are given by the equivalence classes of the roofs $(0, \mathrm{id})$, while an additive inverse of the equivalence class of (f, w) is the class of $(-f, w)$. Hence the category of fractions is preadditive.

Again by the very definition, $Loc : f \mapsto (f, \mathrm{id})$, so Loc is additive. In particular, Loc maps biproduct diagrams to biproduct diagrams. Then since the functor is surjective on objects, $\mathcal{C}[\mathcal{W}^{-1}]$ has biproducts. Any null object of $\mathcal{C}$ also is a null object in $\mathcal{C}[\mathcal{W}^{-1}]$, so the category of fractions is pointed. □

5.4. **Remark.** In an additive category, one can obviously replace the category of fractions axiom (L2) by the easier version

(L2") Given $w \in \mathcal{W}$ and a morphism f such that $fw = 0$, there exists $w' \in \mathcal{W}$ (or $w' \in \mathcal{W}_L$) such that $w'f = 0$.

$$\bullet \overset{w}{\rightsquigarrow} \bullet \xrightarrow{f} \bullet \overset{w'}{\rightsquigarrow} \bullet$$

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